# SOME COMMENTS ON REVERSING ENTROPY AND TIME

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The definition on entropy that we will use in the calculations that follow is the *differential entropy* which measures the entropy corresponding to a continuous probability density function:

$$h(X) = -\int_{\mathbb{R}} f(x) \log f(x) dx. \tag{1}$$

For our first example, we look at how entropy *increases* as a function of time by considering the solution $u(x,t)$ over $\mathbb{R}$ to the one-dimensional diffusion equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad t > 0. \tag{2}$$

To solve this PDE over $-\infty < x < \infty$, we need to specify an initial condition, $u(x,0) = \delta(x)$, which is simply the standard Dirac delta function. The solution to this problem is the well-known heat kernel:

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right). \tag{3}$$

We evaluate $h(X)$ corresponding to $u(x,t)$ above for several values of $t$. So, for example, when $t = 1$, we have:

$$u(x,1) = \frac{e^{-\frac{x^2}{4}}}{2\sqrt{\pi}}. \tag{4}$$

Then, $h(X)$ is found to be:

$$h(X) = -\int_{-\infty}^{\infty} u(x,1) \log u(x,1) dx = \frac{1}{2}\left(1 - \log\left(\frac{1}{4\pi}\right)\right) \approx 1.76551. \tag{5}$$

Similarly, for $t = 2$, we have:

$$u(x,2) = \frac{e^{-\frac{x^2}{8}}}{2\sqrt{2\pi}}. \tag{6}$$

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Then, $h(X)$ is found to be:

$$h(X) = -\int_{-\infty}^{\infty} u(x,2)\log u(x,2)dx = \frac{1}{2}\left(1 - \log\left(\frac{1}{8\pi}\right)\right) \approx 2.11209. \tag{7}$$

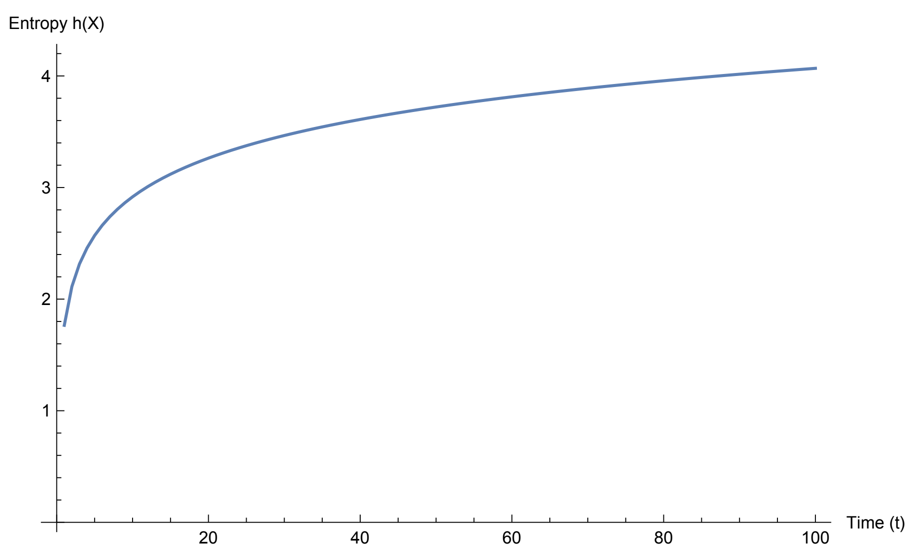
And, just for fun, for $t = 3$, we have:

$$u(x,3) = \frac{e^{-\frac{x^2}{12}}}{2\sqrt{3\pi}}. \tag{8}$$

Then, $h(X)$ is found to be:

$$h(X) = -\int_{-\infty}^{\infty} u(x,3)\log u(x,3)dx = \frac{1}{2}\left(1 - \log\left(\frac{1}{12\pi}\right)\right) \approx 2.31482. \tag{9}$$

Continuing in this fashion as we increase time, i.e., as $t \to +\infty$, we obtain: In fact, we




see from the above plot (and the work above!), that the entropy is related to time in the following fashion:

$$h(t) = 0.5\log(t) + 1.76551. \tag{10}$$

Clearly, as $h'(t) = 0.5/t > 0$ for $t > 0$, we have in a sense "observed" an arrow of time, or at least a positive relationship between an increase in time and an increase in entropy.

An issue with what we have done so far is that the solution that we considered above $u(x,t)$ is not symmetric in time, that is, as $t \to -t$, the solution is fundamentally different:

$$u(x,-t) = \frac{e^{\frac{x^2}{4t}}}{2\sqrt{\pi}\sqrt{-t}}, \tag{11}$$

which is not only invalid from a boundedness sense, but is also imaginary! So, time *irreversibility* is not possible to conclude from what we have done so far.

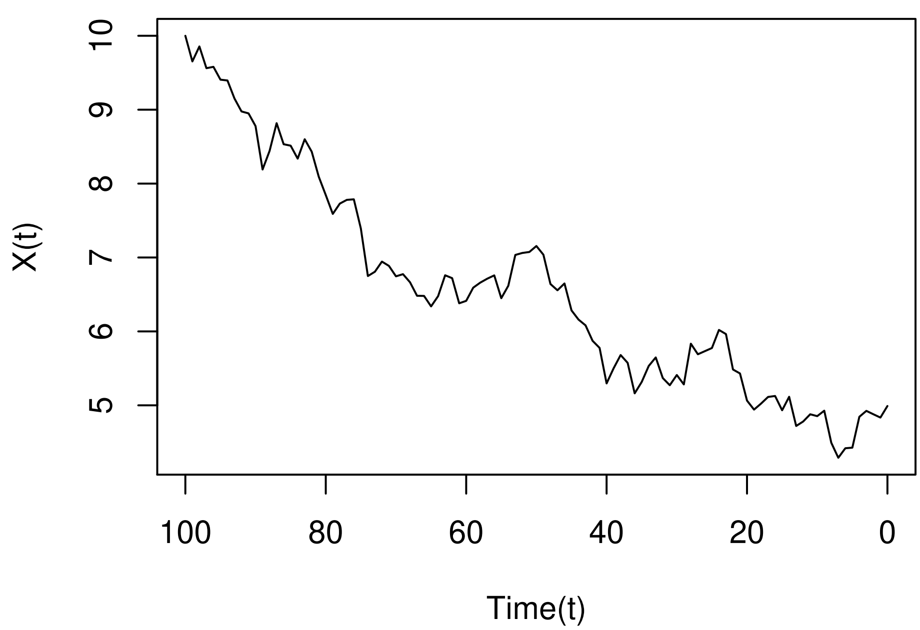
To attempt to derive a time-reversible solution, let us consider a simple arithmetic stochastic differential equation (SDE):

$$dX_t = \mu dt + \sigma dW(t), \tag{12}$$

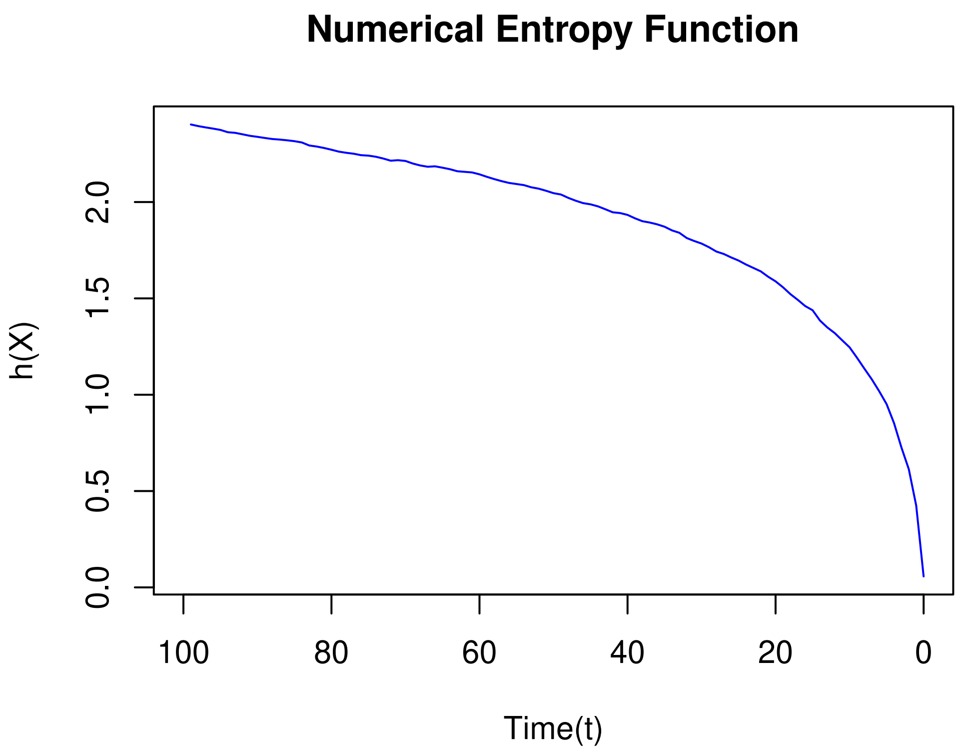
where $W(t)$ is a standard Wiener process. Now, immediately, one will wonder whether or not solving this equation "backwards" is even possible. In general, such problems are not well-posed, and one has to be very careful about establishing conditions of existence and uniqueness of solutions when solving such equations. However, for curiosity-sake, let us assume that we start at a present time denoted by $t = T$, and solve this equation backwards in time to $t = 0$. In general, re-writing the above equation, we have:

$$X_t = -\mu dt - \sigma^2 \sqrt{dt} \mathcal{N}(0, 1) + X_{t+1}. \tag{13}$$

For a numerical solution, I considered 100 backward time steps, set $\mu = 0.1$, $\sigma = 0.5$, and $dt = 1$. Finally, I took that $X_{t=T} = 10$. Performing several Monte Carlo simulations, typical trajectories looked like:



We then evaluated the entropy function $h(X)$ at each backward step in time, and obtained the following results: Interestingly enough, we see the that as we are going "back-




wards" in time, the entropy is decreasing. Of course, as was mentioned in the beginning, which I repeat here, much more mathematically precise arguments have to be made, which I will continue at a later time, when I have more free time to expand on these issues!